CE 297: Problems in the Mathematical Theory of Elasticity: Homework I

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In the following, $z, z_0, z_1, \cdots$ *represent complex numbers,* $\mathbb{C}$ *the complex plane, and* $\mathbb{C}^*$ *the extended complex plane.*

1. Express the ratio z_1/z_2 in the form $a + ib$, where a and b are real.

2. Find the three cube roots of 1. What is the distance between any pair of roots?

3. For any pair of complex numbers $z_1, z_2 \in \mathbb{C}$, prove the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

   Hence extend it via induction to n complex numbers:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

4. Give a geometric interpretation for the multiplication of any (non-zero) complex number z by i. Extend this to multiplication by a complex number $a + ib$ where $a, b \in \mathbb{R}$.

5. Recall that $\mathbb{C}$ has a vector space structure. Show that the Hermitian inner product defined as $\langle z_1, z_2 \rangle := \overline{z_1}\, z_2$ satisfies the following properties

   (a) $\langle z_1, z_2 \rangle = \overline{\langle z_2, z_1 \rangle}$

   (b) $\langle z_1 + z_2, z_3 \rangle = \langle z_1, z_3 \rangle + \langle z_2, z_3 \rangle$
   $\langle z_1, z_2 + z_3 \rangle = \langle z_1, z_2 \rangle + \langle z_1, z_3 \rangle$

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(c) $\langle cz_1, z_2\rangle = \bar{c}\,\langle z_1, z_2\rangle$

$\langle z_1, cz_2\rangle = c\,\langle z_1, z_2\rangle$

Show that $\langle z, z\rangle$ is a non-negative real number, which is 0 only when $z = 0$. What is the norm induced by this inner product $||z|| = \sqrt{\langle z, z\rangle}$ equal to?

6. Show that the angle θ between a pair of complex numbers z_1 and z_2 can be expressed using the Hermitian inner product and induced norm as

$$\cos\theta = \frac{\mathfrak{Re}\,\langle z_1, z_2\rangle}{||z_1||\,||z_2||} = \frac{\mathfrak{Re}\,\langle z_2, z_1\rangle}{||z_2||\,||z_1||}$$

7. Find the radius of convergence of the power series for the exponential

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

8. Find the domain of definition of the rational function

$$Q(z) = \frac{z}{z^6 + 1}$$

9. Sketch the regions defined by the following inequalities, and indicate whether they are open, closed, bounded, compact.
(i) $|3z + 1 - i| < 5$ (ii) $|z| \leq |z + 1|$ (iii) $\mathfrak{Im}(z) <= \pi$ (iv) $0 < |z - i/2| \leq 2$

10. Consider the map

$$w = z + \frac{1}{z}$$

where, as usual, $z = x + iy$, $w = u + iv$. Find the image of points in the upper half of the complex plane ($y > 0$) which are outside the circle $|z| = 1$.

11. Write down the inverse map from the stereographic sphere to the plane, i.e. x, y in the plane, given X, Y, Z on the sphere.

12. Consider the stereographic projection. Find the curves on the sphere which correspond to the lines $\mathfrak{Re}(z) = 0$ and $\mathfrak{Im}(z) = 0$ in the complex plane.

13. Use the $\epsilon - \delta$ definition to show that

$$\lim_{z \to i} z + \frac{1}{z} = 0$$

14. Find $d\sin(z)/dz$ using the definition of a derivative of a complex function as a limit.

15. Show that the function $f(z) = |z|^2$ is nowhere differentiable. Is it continuous?

16. Derive the Cauchy-Riemann equations in polar coordinates, i.e. by considering $f(z) = u(r, \theta) + iv(r, \theta)$. Hence show that if a holomorphic function has constant modulus, it must itself be constant.

17. Given that $u(x, y) = \cosh(x)\cos(y)$ is the real part of a holomorphic function, find the conjugate harmonic function $v(x, y)$, and thus $f(z)$. Is $v(x, y)$ unique?

18. Systematically analyze the multifunction $f(z) = (bz - a)^{\frac{1}{2}}$ where a and b are constants, by doing the following:

    (i) Show that $z = \dfrac{a}{b}$ and ∞ are branch points.

    (ii) Show that in a circuit that *does not* enclose a branch point, the multifunction returns to its original value

    (iii) Introduce branch cuts and discuss the branches of this multifunction.

    (iv) Take any branch, and discuss its holomorphicity in the cut plane. Find the derivative.